A Vector Galerkin's Method Based on E Fields

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***Abstract*—A vector Galerkin's method based on the transverse electric fields (E formulation) is formulated and presented. The general results are applied to a simple step index fiber with large core-cladding index differences.**

I. Introduction

Of the many numerical methods available for yielding guided mode solutions for practical interested dielectric structures of an arbitrary shape, the Bubnov–Galerkin method [1], so-called series expansion method, has been one of the simplest techniques for the simulation and modeling of optical guided-wave devices. The utility of the method depends highly on the careful choice of the basis functions. Various schemes of the basis functions have been proposed and applied to waveguide devices with different geometry and structure [2]–[4]. A thrust of recent research is to extend the present Galerkin method to the approximation of the vectorial wave propagation. Successful implementation of the vector Galerkin's method has been reported on [5].

However, almost all of the published literature mentioned above, except for [4], restricts the solution of the wave equation to a finite domain and chooses the trigonometric function as basis functions. Unfortunately, this kind of basis function does not satisfy the boundary conditions at infinity, a weakness that will present a problem for modes near cutoff, at which time the fields do not decay exponentially. So, not surprisingly, the results of the Galerkin method based on trigonometric functions depend critically on the choice of that finite domain and it is difficult to know how to choose the size of the artificial domain.

As a consequence, we present in this paper an alternative formulation for the vector Galerkin's method with Hermite–Gaussian basis functions. The choice of this kind of basis function has several advantages. This approach not only forms a complete orthonormal set but also satisfies the boundary conditions at infinity. The new method is based on the transverse electric fields (E formulation).

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II. Formulations

The vector wave equations for the electric fields are

$$\nabla^2 \boldsymbol{E} + \nabla\left(\boldsymbol{E} \cdot \frac{\nabla n^2}{n^2}\right) + n^2 k^2 \boldsymbol{E} = 0 \tag{1}$$

where $k = w\sqrt{\epsilon_0 \mu_0}$ is the wave number in free space and $n = {}_n(x, y)$ is the refractive index of the guiding medium. The operator ∇ can be separated into longitude and transverse components,

$$\nabla = \hat{z}\frac{\partial}{\partial z} + \nabla_T. \tag{2}$$

In case a propagating wave of dependence becomes $\exp(-j\beta z)$, the Laplacian operator turns into the form of $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 - \beta^2$. Suppose that the refractive index n varies slowly along the direction of the wave propagation z, that is, $\partial n/\partial z = 0$. The desired coupled equations from the transverse components of (1) can be approximated by

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + (n^2k^2 - \beta^2)E_x + 2\frac{\partial}{\partial x}\left[E_x\frac{\partial \ln(n)}{\partial x} + E_y\frac{\partial \ln(n)}{\partial y}\right] = 0 \tag{3}$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + (n^2k^2 - \beta^2)E_y + 2\frac{\partial}{\partial y}\left[E_x\frac{\partial \ln(n)}{\partial x} + E_y\frac{\partial \ln(n)}{\partial y}\right] = 0. \tag{4}$$

A solution of the coupled equations (3) and (4) can be obtained by expanding Ex and Ey as

$$E_x = \sum_{\mu=0}^{N_x} \sum_{\nu=0}^{N_y} A_{\mu\nu}\Phi_\mu(x)\Phi_\nu(y) \tag{5}$$

$$E_y = \sum_{\mu=0}^{N_x} \sum_{\nu=0}^{N_y} B_{\mu\nu}\Phi_\mu(x)\Phi_\nu(y) \tag{6}$$

where ϕ_μ and ϕ_ν are Hermite–Gaussian functions of order μ, ν,

$$\phi_\mu(x) = \frac{H_\mu(x)\exp(-x^2/2)}{d_\mu} \tag{7}$$

and the same is true with $\phi_\nu(y)$; H_μ is again the Hermite polynomial of order μ. The normalizing constants d_μ and d_ν are chosen to be

$$d_\mu = \pi^{1/4} \cdot \sqrt{\mu!} \cdot 2^{\mu/2} \tag{8a}$$

$$d_\nu = \pi^{1/4} \cdot \sqrt{\nu!} \cdot 2^{\nu/2} \tag{8b}$$

so that the orthogonality condition becomes

$$\int_{-\infty}^{\infty} \frac{H_\mu(x) H_{\mu'}(x) \exp(-x^2)}{d_\mu d_{\mu'}} dx = \delta_{\mu,\mu'} \tag{9}$$

and the same is true with d_ν again.

According to [5], we must remove the derivatives of the logarithm of the refractive index in (3) and (4) using integration by parts, a removal that will naturally dispel the residual terms before the coupled partial differential equations are converted into the desired set of linear equations. Substitute (5) and (6) into (7) and (8), multiply (7) and (8) by $\phi_{\mu'}\phi_{\nu'}$, and integrate the whole space, and we have the following coupled linear equations:

$$\sum_{\mu=0}^{n_x} \sum_{\nu=0}^{n_y} (M_{\mu'\nu',\mu\nu} A_{\mu\nu} + N_{\mu'\nu',\mu\nu} B_{\mu\nu}) = \left(\frac{\beta}{k}\right)^2 A_{\mu'\nu'} \tag{10}$$

$$\sum_{\mu=0}^{n_x} \sum_{\nu=0}^{n_y} (R_{\mu'\nu',\mu\nu} A_{\mu\nu} + S_{\mu'\nu',\mu\nu} B_{\mu\nu}) = \left(\frac{\beta}{k}\right)^2 B_{\mu'\nu'} \tag{11}$$

with the individual matrix elements

$$\begin{aligned} M_{\mu'\nu',\mu\nu} = \frac{1}{k^2} \Big\{ & -(\mu+\nu+1)\delta_{\mu,\mu'}\delta_{\nu,\nu'} \\ & + \frac{\sqrt{(\mu+1)(\mu+2)}}{2} \delta_{\mu+2,\mu'}\delta_{\nu,\nu'} + \frac{\sqrt{\mu(\mu-1)}}{2} \\ & \cdot \delta_{\mu-2,\mu'}\delta_{\nu,\nu'} + \frac{\sqrt{(\nu+1)(\nu+2)}}{2} \delta_{\mu,\mu'}\delta_{\nu+2,\nu'} \\ & + \frac{\sqrt{(\nu(\nu-1)}}{2} \delta_{\mu,\mu'}\delta_{\nu-2,\nu'} + k^2 \\ & \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} n^2(x,y)\phi_\mu(x)\phi_{\mu'}(x)\phi_\nu(y)\phi_{\nu'}(y)\,dx\,dy \\ & + 2 \cdot \frac{1}{d_\mu d_{\mu'} d_\nu d_{\nu'}} \\ & \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \ln(n(x,y)) \cdot \exp(-(x^2+y^2)) \\ & \cdot \big[2x^2 H_\mu(x)H_{\mu'}(x)H_\nu(y)H_{\nu'}(y) - (2\mu'+1) \\ & \cdot H_\mu(x)H_{\mu'}(x)H_\nu(y)H_{\nu'}(y) - 2\mu x H_{\mu-1}(x) \\ & \cdot H_{\mu'}(x)H_\nu(y)H_{\nu'}(y) - 2\mu' x H_\mu(x)H_{\mu'-1}(x) \\ & \cdot H_\nu(y)H_{\nu'}(y) + 4\mu\mu' H_{\mu-1}(x) \\ & \cdot H_{\mu'-1}(x)H_\nu(y)H_{\nu'}(y)\big]\,dx\,dy \Big\} \end{aligned} \tag{12}$$

$$\begin{aligned} N_{\mu'\nu',\mu\nu} = \frac{2}{k^2} \cdot \frac{1}{d_\mu d_{\mu'} d_\nu d_{\nu'}} \cdot & \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \ln(n(x,y)) \\ & \cdot \exp(-(x^2+y^2)) \cdot \big[2xy H_\mu(x) \\ & \cdot H_{\mu'}(x)H_\nu(y)H_{\nu'}(y) \\ & - 2\nu x H_\mu(x)H_{\mu'}(x)H_{\nu-1}(y)H_{\nu'}(y) \\ & - 2\nu' x \cdot H_\mu(x)H_{\mu'}(x)H_\nu(y)H_{\nu'-1}(y) \\ & - 4\mu' y H_\mu(x)H_{\mu'-1}(x)H_\nu(y)H_{\nu'}(y) \\ & + 4\mu'\nu H_\mu(x)H_{\mu'-1}(x)H_{\nu-1}(y)H_{\nu'}(y) \\ & + 4\mu'\nu' H_\mu(x)H_{\mu'-1}(x)H_\nu(y) \\ & \cdot H_{\nu'-1}(y)\big]\,dx\,dy \end{aligned} \tag{13}$$

$$\begin{aligned} R_{\mu'\nu',\mu\nu} = \frac{2}{k^2} \cdot \frac{1}{d_\mu d_{\mu'} d_\nu d_{\nu'}} \cdot & \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \ln(n(x,y)) \\ & \cdot \exp(-(x^2+y^2)) \cdot \big[2xy H_\mu(x) \\ & \cdot H_{\mu'}(x)H_\nu(y)H_{\nu'}(y) - 2\mu y H_{\mu-1}(x) \\ & \cdot H_{\mu'}(x)H_\nu(y)H_{\nu'}(y) - 2\mu' y \\ & \cdot H_\mu(x)H_{\mu'-1}(x)H_y(y)H_{\nu'}(y) \\ & - 4\nu' x H_\mu(x)H_{\mu'}(x)H_\nu(y)H_{\nu'-1}(y) \\ & + 4\mu\nu' H_{\mu-1}(x)H_{\mu'}(x)H_\nu(y)H_{\nu'-1}(y) \\ & + 4\mu'\nu' H_\mu(x)H_{\mu'-1}(x)H_\nu(y) \\ & \cdot H_{\nu'-1}(y)\big]\,dx\,dy \end{aligned} \tag{14}$$

$$\begin{aligned} S_{\mu'\nu',\mu\nu} = \frac{1}{k^2} \Big\{ & -(\mu+\nu+1)\delta_{\mu,\mu'}\delta_{\nu,\nu'} \\ & + \frac{\sqrt{(\mu+1)(\mu+2)}}{2} \delta_{\mu+2,\mu'}\delta_{\nu,\nu'} \\ & + \frac{\sqrt{\mu(\mu-1)}}{2} \cdot \delta_{\mu-2,\mu'}\delta_{\nu,\nu'} \\ & + \frac{\sqrt{(\nu+1)(\nu+2)}}{2} \delta_{\mu,\mu'}\delta_{\nu+2,\nu'} \\ & + \frac{\sqrt{(\nu(\nu-1)}}{2} \delta_{\mu,\mu'}\delta_{\nu-2,\nu'} + k^2 \\ & \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} n^2(x,y)\phi_\mu(x)\phi_{\mu'}(x)\phi_\nu(y)\phi_{\nu'}(y)\,dx\,dy \\ & + 2 \cdot \frac{1}{d_\mu d_{\mu'} d_\nu d_{\nu'}} \\ & \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \ln(n(x,y)) \cdot \exp(-(x^2+y^2)) \\ & \cdot \big[2y^2 H_\mu(x)H_{\mu'}(x)H_\nu(y)H_{\nu'}(y) \\ & -(2\nu'+1)H_\mu(x)H_{\mu'}(x)H_\nu(y)H_{\nu'}(y) \\ & - 2\nu y H_\mu(x)H_{\mu'}(x)H_{\nu-1}(y) \\ & \cdot H_{\nu'}(y) - 2\nu' y H_\mu(x)H_{\mu'}(x)H_\nu(y)H_{\nu'-1}(y) \\ & + 4\nu\nu' H_\mu(x)H_{\mu'}(x)H_{\nu-1}(y)H_{\nu'-1}(y)\big]\,dx\,dy \Big\}. \end{aligned} \tag{15}$$

To be able to combine (10) and (11) into a conventional matrix form, it is necessary to convert the double-index notation into a single-index notation and redefine a vector X consisting of elements $A_{\mu\nu}$ and $B_{\mu\nu}$. The two equation systems (10) and (11) become the standard matrix eigenvalue form

$$CX = \left(\frac{\beta}{k}\right)^2 X \tag{16}$$

with the eigenvalue $(\beta/k)^2$ and eigenvector X.

III. Numerical Applications

To assess and test the accuracy of the vector Galerkin's method based on the E field, we simulated a step-index optical-fiber of circular cross section and compared our results with the known exact solution. The circular fiber has the following parameters: core radius $a = 0.5$ μm, vacuum wavelength $\lambda = 1$ μm, core index $n_1 = 2$, and cladding index $n_2 = 1$. We limited ourselves to the EH_{11} mode, whose exact eigenvalue is $\beta/k = 1.447945$. Using 16 Hermite–Gaussian basis functions in each direction, we computed from the electric vector wave equations the eigenvalue $\beta/k = 1.4543844$, which is different from the exact value by 0.4%. Because the integrals appearing in (12)–(15) are done numerically, the error in the integration may contribute to the uncertainty in the result. As the exact analytical solutions are available in this test case, the size of the enclosed region can be predicted accurately so that the results based on trigonometric basis functions in [5] are better than ours. However, we still believe that Hermite–Gaussian functions are more adequate for the basis functions than trigonometric functions, provided that the test cases are not amenable to exact analysis and the numerical integration in equations (12)–(15) can be avoided.

IV. Conclusions

To find approximate solutions of the vector wave equation to general dielectric optical waveguides, we provide an alternative formulation method in this paper. We tested our method against the exact solution of a circular step-index fiber and found reasonable agreement in β/k to 0.4% with 16 basis functions in each of the two orthogonal directions. Furthermore, it is noticed that the Hermite–Gaussian basis functions can overcome the difficulties in the choice of the adequate size of the enclosed domain. The difficulties inherent in sine-series expansion methods stem from the unsatisfactory boundary conditions at infinity. The unique capacity will help us make more accurate prediction of the modal propagation constant and field distribution in optical waveguides, which are characterized by large index-differences between adjacent regions. Further research should be continued to include the full wave analysis of strongly guiding waveguides comprised of arbitrary materials with any shapes or index profiles.

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References

[1] S. G. Mikhlin and K. L. Smolitskiy, *Approximate Method for Solution of Differential and Integral Equations*. New York: Elsevier, 1967.

[2] J. P. Meunier, J. Pigeon, and J. N. Massot, "A general approach to the numerical determination of modal propagation constants and field distributions of optical fiber," *Opt. Quantum Electron.*, vol. 13, pp. 71–83, 1981.

[3] C. H. Henry and B. H. Verbeek, "Solution of the scalar wave equation for arbitrarily shaped dielectric waveguides by two-dimensional Fourier analysis," *IEEE J. Lightwave Tech.*, vol. 7, pp. 308–313, 1989.

[4] R. L. Gallawa, I. C. Goyal, Yinggang Tu, and A. K. Ghatak, "Optical waveguide modes: an approximate solution using Galerkin's method with Hermite–Gauss basis functions," *IEEE J. Quantum Electron.*, vol. 27, pp. 518–522, 1991.

[5] D. Marcuse, "Solution of the vector wave equation for general dielectric waveguides by the Galerkin method," *IEEE J. Quantum Electron.*, vol. 28, pp. 459–465, 1992.